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Polyakov spin factor, Berry phase and random walks of spinning particles

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Abstract. The path-integral representations for the propagator and the effective action of three-dimensional Euclidean fields with arbitrary spins are found which differ from the analogous path-integrals for scalar particles by the Polyakov's spin factors. For half-integer and integer spins the spin factor coincides with Abelian and non-Abelian Berry phase, respectively.

1. Introduction

Recently, Polyakov has suggested [1] that in three-dimensional spacetime elementary excitations described after the second quantization by Dirac fermion fields could be interpreted as randomly walking spinning particles with torsion. In particular, the propagator $S(x, y; A)$ of three-dimensional Dirac fermions with mass M, interacting with non-Abelian gauge field $A_\mu(x)$ is equal to the following sum over random paths P_{xy} between the points x and y

$$S(x, y; A) \propto \sum_{P_{xy}} \exp(-ML(P_{xy})) \exp(-\mathrm{i}J\Phi(P_{xy})) P \exp\left(\mathrm{i} \int_{P_{xy}} \mathrm{d}x^\mu\, A_\mu(x)\right) \tag{1.1}$$

where $J = \frac{1}{2}$, $L(P_{xy})$ is the length of the path, P-ordered exponential takes into account the interaction of a gauge field with current induced by a particle moving along path P_{xy}, $\Phi(P_{xy})$ is the torsion of path [1] or, equivalently, the abelian Berry phase [2-4], one-dimensional Wess-Zumino-Novikov-Witten term [5]. Later, the above relation has been proven and generalized to higher spacetime dimensions [6-8].

Let us consider the RHS of (1.1) for an arbitrary J and treat it as an amplitude for a spinning particle to go from the point x to y. Then, it turns out that [7]

(i) the parameter J can have only quantized (integer or half-integer) values

$$J = 0, \tfrac{1}{2}, 1, \ldots$$

(ii) the wavefunction of a spinning particle after rotation along some axis with angle 2π is transformed as

$$\psi(x) \to (-1)^{2J} \psi(x)$$

(iii) for integer (or half-integer) J identical spinning particles possess Bose (or Fermi) statistics.

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It is natural to suppose that the parameter J in (1.1) is the spin of spinning particles which after the second quantization are described by quantum fields with spin J. It is this statement that is proved in the present paper. The representation for the effective action and the propagator of three-dimensional Euclidean interacting fields with spin J is found as sums over random paths for a spinning particle with torsion.

The consideration is based on the use of the well known approach [9] in which classical fields with arbitrary spins obey the wave differential equations of first order.

2. Quantum fields with an arbitrary spin

In D-dimensional Euclidean spacetime the components $\varphi_\alpha(p)$ of the classical field with spin J and mass M have to satisfy in the momentum representation the Klein-Gordon equation

$$(p^2+M^2)\varphi_\alpha(p)=0. \tag{2.1}$$

It is well known [9] that the Klein-Gordon equation can be rewritten as a system of linear differential equations of the first order which in the momentum representation has the form

$$((p\cdot\Gamma)+\mathrm{i}M)\phi(p)=0 \tag{2.2}$$

where $\phi(p)$ is a column consisting of fields $\varphi_\alpha(p)$ and their first derivatives, $(p\cdot\Gamma)\equiv p^\mu\Gamma_\mu$ and Γ_μ are some matrices. The action whose variation leads to (2.2) is

$$\mathcal{S}=\int \mathrm{d}^Dp\,\bar{\phi}(-p)((p\cdot\Gamma)+\mathrm{i}M)\phi(p) \tag{2.3}$$

where $\bar{\phi}(p)=\phi^*(p)\Gamma$ is a 'conjugate' field and matrix Γ is a solution of the relation $\Gamma_\mu^+\Gamma^+=\Gamma\Gamma_\mu$.

In the following, we restrict ourselves only to three-dimensional fields. Then, the massive scalar field $\varphi(p)$ is described by (2.2) provided that

$$(p\cdot\Gamma)=\begin{pmatrix}0 & p_1 & p_2 & p_3\\ p_1 & & & \\ p_2 & & 0 & \\ p_3 & & & \end{pmatrix}\equiv\begin{pmatrix}0 & \boldsymbol{p}\\ \boldsymbol{p} & 0\end{pmatrix} \qquad \phi(p)=\begin{pmatrix}\varphi(p)\\ (\mathrm{i}/M)\boldsymbol{p}\varphi(p)\end{pmatrix} \tag{2.4}$$

where $\boldsymbol{p}=(p_1,p_2,p_3)$ is three-dimensional momentum. The classical massive vector field satisfies the Proca equations

$$F_{\mu\nu}(p)=-\mathrm{i}p_\mu A_\nu(p)+\mathrm{i}p_\nu A_\mu(p) \qquad p_\nu F_{\mu\nu}=-\mathrm{i}M^2A_\mu(p) \tag{2.5}$$

which are equivalent to (2.2) for

$$(p\cdot\Gamma)=\begin{pmatrix}0 & -(p\cdot\varepsilon)\\ (p\cdot\varepsilon) & 0\end{pmatrix} \qquad \phi(p)=\begin{pmatrix}A_\mu(p)\\ F_\mu(p)\end{pmatrix} \tag{2.6}$$

where $(p\cdot\varepsilon)_{\mu\rho}=\varepsilon_{\mu\nu\rho}p_\nu$ and $F_\mu(p)=-(1/2M)\varepsilon_{\mu\nu\rho}F_{\nu\rho}$ is the vector dual to the strength tensor. Fields with spin-$\frac{3}{2}$ obey the Rarita-Schwinger equations

$$(\hat{p}+\mathrm{i}M)\psi_\mu(p)=0 \qquad \gamma^\mu\psi_\mu(p)=0 \tag{2.7}$$

where $\hat{p} \equiv p_\mu \gamma^\mu$ and $\gamma^\mu = \sigma^\mu$ are Dirac matrices coinciding in $D=3$ spacetime with Pauli matrices. To replace these equations by (2.2), one has to put

$$(p \cdot \Gamma)_{\mu\nu} = \tfrac{1}{2}(\delta_{\mu\nu}\hat{p} - \sigma_\mu p_\nu - \mathrm{i}\varepsilon_{\mu\nu\rho}p_\rho) \qquad \phi(p) \equiv \psi_\mu(p) \tag{2.8}$$

where spinor indices of the Γ^μ-matrices are imposed.

For fields with an arbitrary spin the matrices Γ^μ are restricted by the condition that all components of the field $\phi(p)$ have to satisfy the Klein–Gordon equation (2.1). For massive fields ($M \neq 0$) this condition is fulfilled if one imposes the following equation on matrices Γ^μ †

$$(p \cdot \Gamma)^{K+2} = p^2 (p \cdot \Gamma)^K \tag{2.9}$$

where K is a non-negative integer number and p_μ is an arbitrary three-vector. At $K=0$ the solutions of the equation are Dirac matrices. At $K=1$ the equation has non-trivial Hermitian solutions different from Dirac matrices provided that Γ^μ-matrices are degenerated‡

$$K=1 \qquad (p\cdot\Gamma)^3 = p^2(p\cdot\Gamma) \qquad \det(p\cdot\Gamma)=0 \qquad (p\cdot\Gamma)^+ = (p\cdot\Gamma). \tag{2.10}$$

At $K \geqslant 2$ (2.9) has non-trivial solutions different from the previous ones provided that Γ^μ-matrices are degenerated and non-Hermitian

$$K \geqslant 2 \qquad \det(p\cdot\Gamma)=0 \qquad (p\cdot\Gamma)^+ \neq (p\cdot\Gamma).$$

It can be easily verified that matrices (2.4) and (2.6) obey (2.10) but matrices (2.8) correspond to the $K=2$ case of (2.9).

The parameter K entering into (2.9) has a simple meaning [9]. It determines the spin content of the field $\phi(p)$. In general, $\phi(p)$ can be decomposed into a sum of two fields with spins $J=(1+K)/2$ and $|(1-K)/2|$§. For example, at $K=0$, (2.2) and (2.9) describe Dirac fermions with spin $J=\frac{1}{2}$; at $K=1$ one has $J=0$, 1 and the field $\phi(p)$ is a sum of a scalar and a vector field (a scalar field and its derivatives in (2.4) or a vector field and vector dual to the strength tensor in (2.6)).

Thus, the theory (2.3) with Γ^μ-matrices (2.9) gives us an equivalent description of classical fields with an arbitrary spin.

After quantization of the theory (2.3) one defines the propagator of a field $\phi(p)$ as a solution of the following equation

$$((p\cdot\Gamma)+\mathrm{i}M)S_0(P)=1 \tag{2.11}$$

whose solution is [9]

$$S_0(p) = \frac{1}{p^2+M^2}\left((p\cdot\Gamma) - \mathrm{i}M + \sum_{n=1}^{K}\left(\frac{\mathrm{i}}{M}\right)^n (p\cdot\Gamma)^{n-1}((p\cdot\Gamma)^2 - p^2)\right) \tag{2.12}$$

where the parameter K has appeared before in (2.9). We notice that, first, the propagator has a dangerous ultraviolet asymptotics $S_0(p) \xrightarrow{p\to\infty} p^{-1+K}$ and, second, it has a singularity in the limit $M \to 0$.

† It is not the only possible equation. There is another one [9] called 'multi-mass' equation.

‡ Moreover, the Lorentz covariance of (2.2) enables us to replace these equations by the Cemmer–Duffin ones [9]

$$\Gamma^\mu\Gamma^\nu\Gamma^\rho + \Gamma^\rho\Gamma^\nu\Gamma^\mu = \delta^{\mu\nu}\Gamma^\rho + \delta^{\nu\rho}\Gamma^\mu.$$

§ This value appears if the field $\phi(p)$ is transformed in accordance with the reducible representation of the SO(3) group.

To understand the reason of this effect one notes that (2.11) takes into account the propagation of elementary excitations whose wavefunctions are eigenstates of the operator $(p\cdot\Gamma)$ and $(p\cdot\Gamma)^+$

$$(p\cdot\Gamma)|\lambda\rangle=\lambda|\lambda\rangle. \tag{2.13}$$

It follows from (2.9) that the eigenvalues λ are the solutions of the equation $\lambda^K(\lambda^2-p^2)=0$ or

$$\lambda=0,+p,-p. \tag{2.14}$$

The eigenvalues are in general degenerated and the value $\lambda=0$ appears for $K\geqslant 1$. Then, the classical equations of motion (2.2) can be rewritten as $(\lambda(p)+\mathrm{i}M)|\lambda\rangle=0$ or

$$(\lambda^2(p)+M^2)|\lambda\rangle=0.$$

We conclude that for $M\neq 0$ among all the eigenstates of the operator $(p\cdot\Gamma)$ only those corresponding to non-zero λ satisfy the Klein–Gordon equation (2.1). The eigenstates $|\lambda=0\rangle$ called zero modes do not obey (2.1). Nevertheless, after quantization the zero modes contribute to the propagator. As a result, the propagator (2.12) has both dangerous ultraviolet asymptotics and singular limit $M\to 0$.

Thus, to avoid these difficulties, one imposes the additional constraints on the field $\phi(p)$ forbidding the propagation of zero modes

$$\Pi_0(p)\phi(p)=0 \tag{2.15}$$

where $\Pi_0(p)$ is the projector onto degenerated zero modes of the operators $(p\cdot\Gamma)$ and $(p\cdot\Gamma)^+$†.

To find the explicit form of the projector $\Pi_0(p)$ one notes that although the operator $(p\cdot\Gamma)$ is not Hermitian for $K\geqslant 2$ the operators $(p\cdot\Gamma)^K/p^K$ and $(p\cdot\Gamma)^{K+1}/p^{K+1}$ obey equations like $A^3=A$ and in accordance with (2.10) can be chosen to be Hermitian. Hence, using (2.13) and (2.14) these operators are expressed as

$$\begin{aligned}(p\cdot\Gamma)^K&=p^K\Pi_+(p)+(-p)^K\Pi_-(p)\\(p\cdot\Gamma)^{K+1}&=p^{K+1}\Pi_+(p)+(-p)^{K+1}\Pi_-(p)\end{aligned} \tag{2.16}$$

where $\Pi_\pm(p)$ are projectors onto eigenstates corresponding to the degenerate eigenvalues $\lambda=\pm p$. The Hermiticity of the above operators implies that non-zero modes of the operators $(p\cdot\Gamma)$ and $(p\cdot\Gamma)^+$ coincide and $\Pi_+(p)\Pi_-(p)=\Pi_\pm(p)\Pi_0(p)=0$.

Adding the 'completeness' relation

$$\mathbf{1}=\Pi_+(p)+\Pi_-(p)+\Pi_0(p)$$

to (2.16), one finds

$$\begin{aligned}&\Pi_+(p)=\tfrac{1}{2}((e\cdot\Gamma)^K+(e\cdot\Gamma)^{K+1}) \qquad \Pi_-(p)=\Pi_+(-p),\\&\Pi_0(p)=1-(e\cdot\Gamma)^K\frac{1+(-1)^K}{2}-(e\cdot\Gamma)^{K+1}\frac{1-(-1)^K}{2} \qquad p=pe.\end{aligned} \tag{2.17}$$

The operator $(p\cdot\Gamma)$ can be decomposed as follows:

$$(p\cdot\Gamma)=p\Pi_+(p)-p\Pi_-(p)+\mathscr{P} \tag{2.18}$$

† For $K\geqslant 2$ the operator $(p\cdot\Gamma)$ is not Hermitian and zero modes of $(p\cdot\Gamma)$ and $(p\cdot\Gamma)^+$ are different and form, as will be proven below, the orthonormal basis in the subspace of zero modes.

where $\mathscr{P}$ is a non-Hermitian operator acting in the subspace of zero modes:

$$\mathscr{P}\Pi_\pm(p)=\Pi_\pm(p)\mathscr{P}=0 \qquad \mathscr{P}\Pi_0(p)=\Pi_0(p)\mathscr{P}=\mathscr{P}$$

and (2.16) implies that $\mathscr{P}^K=0$.

To understand the meaning of zero modes one has to consider the action (2.3) in the limit $M\to 0$. In this limit the properties of the theory are changed since the action (2.3) becomes invariant under the following gauge transformations

$$\phi(p)\to\phi(p)+\Pi_0(p)\mathscr{P}^{K-1}\xi(p) \tag{2.19}$$

with $\xi(p)$ being an arbitrary state. Then, under quantization of the massless theory the gauge condition (2.15) must be put on the field $\phi(p)$ to single out the only element among all the gauge equivalent fields (2.19). One notices that these transformations affect only zero modes of field $\phi(p)$. Thus, zero modes describe degrees of freedom of the quantum field $\phi(p)$ which become gauge in the limit of vanishing mass M.

After imposing the gauge condition (2.15) one gets the following equation for the propagator:

$$\left((p\cdot\Gamma)+\mathrm{i}M+\frac{1}{\alpha}\Pi_0(p)\right)S(p)=1 \qquad \alpha\to 0. \tag{2.20}$$

The solution to this equation is

$$S(p)=(\Pi_+(p)+\Pi_-(p))S_0(p)=(\Pi_+(p)+\Pi_-(p))\frac{(p\cdot\Gamma)-\mathrm{i}M}{p^2+M^2}=\frac{\Pi_+(p)}{p+\mathrm{i}M}-\frac{\Pi_-(p)}{p-\mathrm{i}M}$$

where $S_0(p)$ was defined in (2.12) and the identity $\Pi_\pm(p)((p\cdot\Gamma)^2-p^2)=0$ following from (2.9) and (2.17) was used. After substitution of (2.17) the propagator is given for odd K by

$$S(p)=\frac{p(e\cdot\Gamma)^K-\mathrm{i}M(e\cdot\Gamma)^{K+1}}{p^2+M^2} \qquad p=pe \tag{2.21}$$

and for even K by

$$S(p)=\frac{p(e\cdot\Gamma)^{K+1}-\mathrm{i}M(e\cdot\Gamma)^K}{p^2+M^2} \qquad p=pe. \tag{2.22}$$

Thus, the propagator (2.21) and (2.22) of the physical polarizations has well defined limit $M\to 0$ and ultraviolet asymptotics $S(p)\xrightarrow{p\to\infty}1/p$.

Let us consider the explicit form of the gauge condition (2.15) for scalar and vector fields. For the scalar field we denote the components of the field $\phi(p)$ as

$$\phi(p)=\begin{pmatrix}\varphi(p)\\ \varphi_\mu(p)\end{pmatrix}.$$

Combining (2.4), (2.17) and (2.19) one finds that at $M=0$ the gauge ambiguity reduces to

$$\phi_\mu(p)\to\phi_\mu(p)+\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\xi_\nu(p).$$

The gauge condition (2.15) becomes

$$\varphi_\mu(p)=p_\mu\frac{1}{p^2}p_\nu\varphi_\nu(p)$$

and it implies that quantum field $\varphi_\mu(p)$ has to be longitudinally polarized. This condition is fulfilled identically in the classical theory (2.4) with $M \neq 0$ where $\varphi_\mu(p)$ is proportional to derivative of the scalar field $\varphi(p)$. The propagator of the field $\phi(p)$ is equal in this gauge to

$$S(p)=\frac{1}{p^2+M^2}\begin{pmatrix} -\mathrm{i}M & p \\ p & -\mathrm{i}M(p\otimes p)/p^2 \end{pmatrix} \tag{2.23}$$

and up to unessential factors the element $S_{11}(p)$ of this matrix coincides with scalar propagator.

For a massive vector field one chooses the field $\phi(p)$ in the form

$$\phi(p)=\begin{pmatrix} A_\mu(p) \\ F_\mu(p) \end{pmatrix}.$$

After substitution of matrices (2.6) into (2.17) and (2.19) we get that in the massless theory gauge transformations have the well known form

$$A_\mu(p)\to A_\mu(p)+p_\mu\xi(p) \qquad F_\mu(p)\to F_\mu(p)+p_\mu\xi'(p)$$

and the gauge condition (2.15) leads to the Lorentz gauge

$$p_\mu A_\mu(p)=p_\mu F_\mu(p)=0.$$

As in the previous case the same condition follows from classical equations of motion (2.5). The propagator of the field $\phi(p)$ in this gauge is given by

$$S(p)=\frac{1}{p^2+M^2}\begin{pmatrix} -\mathrm{i}(M/p^2)(p^2I-p\otimes p) & (p\cdot\varepsilon) \\ (p\cdot\varepsilon) & -\mathrm{i}(M/p^2)(p^2I-p\otimes p) \end{pmatrix}. \tag{2.24}$$

It is interesting to recognize that the element $S_{11}(p)/M$ of this matrix is equal to the propagator of the field $A_\mu(p)$ in the Stueckelberg theory [10]

$$\mathscr{L}=\tfrac{1}{4}(\partial_\mu A_\nu-\partial_\nu A_\mu)^2+\tfrac{1}{2}M^2A_\mu^2+\frac{1}{2\alpha}(\partial_\mu A_\mu)^2 \qquad \alpha\to 0.$$

At the same time, in the theory without a gauge fixing term the propagator of the vector field is

$$\frac{1}{p^2+M^2}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{M^2}\right)$$

and it has all the problems pointed out before.

3. Propagator and effective action as sums over random paths

In the previous section we dealt with free fields. Let us generalize the consideration of interacting fields as follows. We introduce the interaction of the field $\phi(p)$ with nonabelian gauge field $A_\mu(x)$ by replacing the ordinary derivatives of $\phi(x)$ by the covariant ones (or in the momentum representation $p_\mu\to p_\mu+A_\mu(x)$) in the equations of motion (2.2), in the action (2.3) and in the gauge condition (2.15). However, the covariant derivatives do not commutate ($[D_\mu, D_\nu]=-\mathrm{i}F_{\mu\nu}$) and we have to take care of the ordering of covariant derivatives in all above relations. It is possible to find the proper ordering only in the special cases $K=0$ and $K=1$ but for higher $K\geqslant 2$ we come across the known consistency problem for interacting higher spin fields. To avoid this difficulty one assumes that $A_\mu(x)$ is a pure gauge field. The action of thus defined theories is invariant under gauge transformations

$$A_\mu(x) \to U^+(x)(A_\mu(x) + \mathrm{i}\partial_\mu)U(x), \qquad \phi(x) \to U(x)\phi(x).$$

The propagator and the effective action of interacting fields of an arbitrary spin $\phi(p)$ in the x-representation are defined analogously to (2.20)

$$S(x, y; A) = \langle x|(H(\pi) + \mathrm{i}M)^{-1}|y\rangle = -\mathrm{i}\int_0^\infty \mathrm{d}T\, \mathrm{e}^{-TM}\langle x|\mathrm{e}^{\mathrm{i}TH(\pi)}|y\rangle \tag{3.1}$$

and

$$W[A] = -\mathrm{Tr}\log(H(\pi) + \mathrm{i}M) = -\int_0^\infty \frac{\mathrm{d}T}{T}\mathrm{e}^{-TM}\,\mathrm{Tr}\int \mathrm{d}^3x\langle x|\,\mathrm{e}^{\mathrm{i}TH(\pi)}|x\rangle \tag{3.2}$$

where

$$H(\pi) = (\pi\cdot\Gamma) + \frac{1}{\alpha}\Pi_0(\pi) \qquad \pi_\mu = p_\mu + A_\mu(x) \equiv \mathrm{i}\partial_\mu + A_\mu(x) \qquad \alpha \to 0. \tag{3.3}$$

The expression $\langle x|\,\mathrm{e}^{\mathrm{i}TH(\pi)}|y\rangle$ entering into (3.1) and (3.2) can be treated [11] as a matrix element of the evolution operator for a spinning particle with Hamiltonian H. Then, $\langle x|\,\mathrm{e}^{\mathrm{i}TH}|y\rangle$ being an amplitude for a particle to go from point x to y at a proper time T is equal, following Feynman [12], to a path integral over the phase space of a particle [11, 13, 14]

$$\langle x|\,\mathrm{e}^{\mathrm{i}TH(p+A(x))}|y\rangle = \int_y^x \mathscr{D}x_\mu\, P\exp\left(\mathrm{i}\int_{P_{xy}} \mathrm{d}x^\mu A_\mu(x)\right)\mathscr{M}[P_{xy}] \tag{3.4}$$

where the integration is performed over all x-paths P_{xy} between the points x and y, and the integral over unrestricted momentum paths is factorized into

$$\mathscr{M}[P_{xy}] = \int \mathscr{D}p_\mu \exp\left(-\mathrm{i}\int_0^T \mathrm{d}t(p\cdot\dot{x})\right)T\exp\left(\mathrm{i}\int_0^T \mathrm{d}t\, H(p)\right). \tag{3.5}$$

The matrix $H(p)$ is defined as

$$H(p) = (p\cdot\Gamma) + \frac{1}{\alpha}\Pi_0(p) = p\Pi_+(p) - p\Pi_-(p) + \frac{1}{\alpha}(\Pi_0(p) + \alpha\mathscr{P}) \tag{3.6}$$

where (2.18) and (3.3) are used.

To calculate the path integral in (3.5) we have to regularize the integrand for large values of momenta by inserting the ultraviolet cut-off factor [13] $\exp(-\int_0^T \mathrm{d}t\,\varepsilon(t)p(t))$, $\varepsilon(t) \to 0$ into the RHS of (3.5). Then, dividing time interval $[0, T]$ into N equal pieces the path integral $\mathscr{M}[P_{xy}]$ is approximated as

$$\begin{aligned} \mathscr{M}[P_{xy}] &= \lim_{N\to\infty}\prod_{i=1}^{N}\mathscr{M}[\dot{x}(\mathrm{i}\tau)] \\ \mathscr{M}[\dot{x}] &= \int \mathrm{d}^3p\,\exp(-\mathrm{i}(p\cdot\dot{x})\tau + \mathrm{i}H(p)\tau - \varepsilon p\tau) \qquad \tau = \frac{T}{N}. \end{aligned} \tag{3.7}$$

Using the identity $e^{iH(p)\tau} = \Pi_+(p)\, e^{ip\tau} + \Pi_-(p)\, e^{-ip\tau} + \Pi_0(p)(1+\mathcal{O}(\tau))\, e^{i\tau/\alpha}$ following from (3.6) one finds in the limit $\alpha, \varepsilon \to 0$ up to unessential factor

$$\mathcal{M}[\dot{x}] = \delta(1-\dot{x}^2)\Pi_+(\dot{x}) \equiv \delta(1-\dot{x}^2) \sum_{a=1}^{r} |\dot{x}, a\rangle\langle\dot{x}, a| \tag{3.8}$$

where $|\dot{x}, a\rangle$ are the eigenstates of operator $(\dot{x}\cdot\Gamma)$

$$(\dot{x}\cdot\Gamma)|\dot{x}, a\rangle = |\dot{x}, a\rangle \qquad a = 1, \ldots, r \tag{3.9}$$

corresponding to the degenerated eigenvalue $\lambda = |\dot{x}| = 1$ and r is the degree of degeneracy of this eigenvalue

$$r = \operatorname{Tr}\Pi_+(\dot{x}) = \tfrac{1}{2}\operatorname{Tr}((\dot{x}\cdot\Gamma)^K + (\dot{x}\cdot\Gamma)^{K+1}). \tag{3.10}$$

After substitution of (3.8) into eq. (3.7) we obtain

$$\mathcal{M}[P_{xy}] = \delta(1-\dot{x}^2) \prod_{t\in[0,T]} \Pi_+(\dot{x}(t))$$

and the infinite product of projectors in the RHS is known as the Polyakov spin factor.

One starts the evaluation of the spin factor by finding two equivalent representations for it. Let us consider two neighbouring factors $\Pi_+(\dot{x}(t+\delta t))\Pi_+(\dot{x}(t))$ in the infinite product of projectors. After substitution of the decomposition $\Pi_+(\dot{x}) = \Sigma_{a=1}^{r}|\dot{x}, a\rangle\langle\dot{x}, a|$ the product $\Pi_+(\dot{x}(t+\delta t))\Pi_+(\dot{x}(t))$ reduces to the following scalar product

$$\langle\dot{x}(t+\delta t), a|\dot{x}(t), b\rangle = \delta_{ab} - \delta t\left\langle \dot{x}(t), a\left|\frac{d}{dt}\right|\dot{x}(t), b\right\rangle + \mathcal{O}(\delta t^2).$$

As a consequence, for the infinite product of projectors one gets

$$\mathcal{M}[P_{xy}] = \delta(1-\dot{x}^2)|\dot{x}(T), a\rangle\langle\dot{x}(0), b|[T\exp(-i\Phi(P_{xy}))]_{ab} \tag{3.11}$$

where the notation is introduced for a matrix of an order of r

$$\Phi_{ab}(P_{xy}) = -i\int_0^T dt\left\langle\dot{x}, a\left|\frac{d}{dt}\right|\dot{x}, b\right\rangle = \int_{P_{xy}} d\dot{x}^\mu\, \mathcal{A}_\mu^{ab}(\dot{x}) \tag{3.12}$$

depending on the path P_{xy} and known as the non-Abelian Berry phase [2-4]. Here, in the second expression integration is performed along the path $\{\dot{x}_\mu(t), t\in[0, T]\}$ on the sphere S^2 and the vector field

$$\mathcal{A}_\mu^{ab}(\dot{x}) = -i\left\langle\dot{x}, a\left|\frac{\partial}{\partial\dot{x}_\mu}\right|\dot{x}, b\right\rangle \tag{3.13}$$

is the Berry connection.

The representation (3.11) possesses non-Abelian gauge invariance. The origin of this invariance is the following [4]. There is an ambiguity in the definition (3.9) of the eigenstates: $|\dot{x}, a\rangle \to V_{ba}|\dot{x}, b\rangle$, where $V^+V = 1$ and matrices V belong to $U(r)$ group with r being defined in (3.10). Under this transformation the LHS of (3.11) is unchanged but the Berry connection is transformed as non-Abelian gauge field: $\mathcal{A}_\mu(\dot{x}) \to V^+(\mathcal{A}_\mu(\dot{x}) - i\partial/\partial\dot{x}_\mu)V$. One has the following relation

$$-i\frac{\partial}{\partial\dot{x}_\mu}|\dot{x}, a\rangle = \mathcal{A}_\mu^{ba}(\dot{x})|\dot{x}, b\rangle + \mathcal{B}_\mu^{\alpha a}(\dot{x})|\dot{x}, \alpha\rangle$$

where $\mathcal{B}^{\alpha a}(\dot{x})$ is non-Hermitian vector field and $|\dot{x}, \alpha\rangle$ are the eigenstates of the operator $(\dot{x}\cdot\Gamma)/|\dot{x}|$ corresponding to the remaining eigenvalues $\lambda_\alpha = 0$ and $\lambda_\alpha = -1$. It implies that derivative of the state $|\dot{x}, a\rangle$ does not belong to the subspace of the eigenstates $\{|\dot{x}, a\rangle, a = 1, \ldots, r\}$. Using this property one finds the strength of the Berry connection as

$$\mathscr{F}^{ab}_{\mu\nu}(\dot{x}) = \partial_\mu \mathscr{A}^{ab}_\nu - \partial_\nu \mathscr{A}^{ab}_\mu + \mathrm{i}[\mathscr{A}_\mu, \mathscr{A}_\nu]^{ab} = -\mathrm{i}((\mathscr{B}^+_\mu)^{a\alpha}\mathscr{B}^{\alpha b}_\nu - (\mathscr{B}^+_\nu)^{a\alpha}\mathscr{B}^{\alpha b}_\mu).$$

To evaluate the field $\mathscr{B}^{\alpha a}_\mu(\dot{x})$ we rewrite (3.9) as $(\dot{x}\cdot\Gamma)|\dot{x}, a\rangle = |\dot{x}||\dot{x}, a\rangle$ differentiate it both sides over $\dot{x}_\mu$ and multiply by the state $\langle\dot{x}, \alpha|$. Then one obtains

$$\mathscr{B}^{\alpha a}_\mu(\dot{x}) = -\mathrm{i}\left\langle \dot{x}, \alpha \left| \frac{\partial}{\partial \dot{x}_\mu} \right| \dot{x}, a \right\rangle = -\frac{\mathrm{i}}{(1-\lambda_\alpha)|\dot{x}|}\langle \dot{x}, \alpha|\Gamma_\mu|\dot{x}, a\rangle$$

and this expression is valid for arbitrary vectors $\dot{x}_\mu$. Thus, the field strength is expressed as

$$\mathscr{F}^{ab}_{\mu\nu}(\dot{x}) = -\frac{\mathrm{i}}{4\dot{x}^2}\langle \dot{x}, a|\Gamma_\mu(4\Pi_0(\dot{x}) + \Pi_-(\dot{x}))\Gamma_\nu - \Gamma_\nu(4\Pi_0(\dot{x}) + \Pi_-(\dot{x}))\Gamma_\mu|\dot{x}, b\rangle \tag{3.14}$$

where the projector $\Pi_0(\dot{x})$ onto zero modes appears here only for $K \geq 1$. It will be demonstrated at the end of section 4 that, with the explicit form of Γ_μ matrices being taken into account, (3.13) and (3.14) describe field created by monopole with charge equal to spin.

Another representation for the spin factor we find using the following relation proven in the appendix

$$\Pi_+(\dot{x}(t+\delta t))\Pi_+(\dot{x}(t)) = (1 - \mathrm{i}\Sigma^\mu \varepsilon_{\mu\nu\rho}\dot{x}_\nu\ddot{x}_\rho\delta t)\Pi_+(\dot{x}(t)) + \mathscr{O}(\delta t^2) \tag{3.15}$$

where Σ_μ are the generators of the reducible representation of SO(3) group in accordance with which the field $\phi(x)$ is transformed. Then, the infinite product of projectors is replaced by

$$\mathscr{M}[P_{xy}] = \delta(1-\dot{x}^2)\,T\exp\left(-\mathrm{i}\int_0^T \mathrm{d}t\, \Sigma^\mu \varepsilon_{\mu\nu\rho}\dot{x}_\nu\ddot{x}_\rho\right)\Pi_+(\dot{x}(0)).$$

For spinning particles with Hamiltonian (3.3) matrix Σ^μ is equal to the spin part of the angular momentum $M_\mu = \varepsilon_{\mu\nu\rho}x_\nu\pi_\rho + \Sigma_\mu$. Therefore, the path-ordered exponential in this expression is equal to a product of infinitesimal rotations of the wavefunction of a spinning particle under its motion along path P_{xy}. The explicit form of the spin matrices at $K=0$ and $K=1$ is $\Sigma_\mu = i\varepsilon_{\mu\nu\rho}[\Gamma_\nu, \Gamma_\rho]$ but the analogous expressions for $K \geq 2$ are more complicated due to non-Hermiticity of Γ^μ matrices. Multiplying both the sides of (3.11) and the last equation by state $|\dot{x}(0), b\rangle$ one finds the relation

$$T\exp\left(-\mathrm{i}\int_0^T \mathrm{d}t\,\Sigma^\mu\varepsilon_{\mu\nu\rho}\dot{x}_\nu\ddot{x}_\rho\right)|\dot{x}(0), b\rangle = |\dot{x}(T), a\rangle[T\exp(-\mathrm{i}\Phi(P_{xy}))]_{ab}$$

which admits the following interpretation. The state $|\dot{x}(0), b\rangle$ describing the spinning particle with momentum $p_\mu = -\mathrm{i}\dot{x}_\mu M$ and mass M acquires non-trivial Berry phase after parallel transport along path P_{xy} from point $x(0) = y$ to $x(T) = x$.

Finally, combining (3.4) and (3.11) one obtains the following representations for the propagator and the effective action

$$S(x, y; A) = -\mathrm{i}\int_0^\infty \mathrm{d}T\,\mathrm{e}^{-TM}\int_y^x \mathscr{D}x_\mu\,\delta(1-\dot{x}^2)P\exp\left(\mathrm{i}\int_{P_{xy}}\mathrm{d}x^\mu A_\mu(x)\right)$$
$$\times|\dot{x}(T), a\rangle\langle\dot{x}(0), b|[T\exp(-\mathrm{i}\Phi(P_{xy}))]_{ab}$$

and

$$W[A] = -\int_0^\infty \frac{dT}{T} e^{-TM} \int d^3x \int_x^x \mathcal{D}x_\mu \delta(1-\dot{x}^2) \operatorname{Tr} P \exp\left(i \oint_{P_{xx}} dx^\mu A_\mu(x)\right) \times \operatorname{Tr}[T \exp(-i\Phi(P_{xx}))]. \quad (3.16)$$

The spin factor $\Phi(P)$ and the states $|\dot{x}, a\rangle$ are the only undetermined quantities in these relations.

4. The evaluation of the spin factor

The spin factor $\Phi(P)$ and the states $|\dot{x}, a\rangle$ depend on the spin content of the quantum field $\phi(p)$ and can be found by solving eq. (3.9) with definition (2.9) of Γ^μ-matrices for an arbitrary K.

Spin $J=0$. For the scalar field one gets from (2.4) and (3.10) that $r=1$ and (3.9) is replaced by

$$\begin{pmatrix} 0 & \dot{x} \\ \dot{x} & 0 \end{pmatrix} |\dot{x}\rangle = |\dot{x}\rangle.$$

The normalized solution of this equation is

$$|\dot{x}\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \dot{x} \end{pmatrix}.$$

Hence, the spin factor (3.12) for the scalar field is equal to

$$\Phi_{J=0}(P) = -i \int_0^T dt \left\langle \dot{x} \left| \frac{d}{dt} \right| \dot{x} \right\rangle = -\frac{i}{2} \int_0^T dt (\dot{x} \cdot \ddot{x}) = 0$$

since $\dot{x}^2 = 1$.

Spin $J=\frac{1}{2}$. For Dirac fermions one has $r=1$; Γ^μ are Pauli matrices and the solution of equation (3.9) is well-known as a coherent state for the $SU(2)$ group

$$|\dot{x}\rangle = \begin{pmatrix} \cos\theta/2 \\ e^{i\varphi} \sin\theta/2 \end{pmatrix} \quad (4.1)$$

where $\dot{x} = (\sin\theta\cos\varphi, \sin\theta\,\varphi, \cos\theta)$. The spin factor for Dirac fermions is equal to

$$\Phi_{J=1/2}(P) = -i \int_0^T dt \left\langle \dot{x} \left| \frac{d}{dt} \right| \dot{x} \right\rangle = \tfrac{1}{2} \int_0^T dt\, \dot{\varphi}(1-\cos\theta) = \tfrac{1}{2} \int_0^T dt\, C(t)$$

with $C(t)$ being the torsion of the path.

Spin $J=1$. For the vector field one finds from (2.6) and (3.10) that $r=2$ and, hence, the spin factor $\Phi(P)$ is a matrix of second order. With the solution of (3.9) chosen in the form

$$|\dot{x}, a\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}$$

equation (3.9) can be rewritten as

$$\begin{pmatrix} 0 & -(\dot{x}\cdot\varepsilon) \\ (\dot{x}\cdot\varepsilon) & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} = \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \tag{4.2}$$

or in the vector notation

$$[\dot{x}\times e_1] = e_2 \qquad [e_2\times\dot{x}] = e_1 . \tag{4.3}$$

The solutions of these equations are two real orthogonal unit vectors e_1 and e_2 together with tangent vector $\dot{x}$ forming the right oriented basis in the three-dimensional space

$$(e_i\cdot e_j) = \delta_{ij} \qquad (e_i\cdot\dot{x}) = 0.$$

As a result, two orthonormal eigenstates $|\dot{x}, a\rangle$ are

$$|\dot{x}, 1\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} \qquad |\dot{x}, 2\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} -e_2 \\ e_1 \end{pmatrix}. \tag{4.4}$$

Any other solutions of (4.2) can be obtained from these ones after rotation of the vectors e_1 and e_2 in the plane normal to the vector $\dot{x}$. After substitution of (4.4) into (3.12) we obtain for the spin factor

$$\Phi_{J=1}(P) = -\tfrac{1}{2}\sigma_2 \int_0^T \mathrm{d}t(e_1\dot{e}_2 - e_2\dot{e}_1)$$

where

$$\sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

is the Pauli matrix. To understand the geometrical meaning of the integral, one uses the Frene equations

$$\ddot{x} = b_i e_i \qquad \dot{e}_i = -C\varepsilon_{ij}e_j - b_i\dot{x}$$

from which it follows that the spin factor for the vector field is proportional to the torsion of the path

$$\Phi_{J=1}(P) = -\sigma_2 \int_0^T \mathrm{d}t\, C(t).$$

Spin $J=\frac{3}{2}$. For the spin-$\frac{3}{2}$ field (2.8) and (3.10) imply that $r=1$. Equation (3.9) is replaced by

$$(\dot{x}\cdot\Gamma)_{\mu\nu}|\dot{x};\nu\rangle = |\dot{x};\mu\rangle$$

with the matrices Γ^μ being defined in (2.8). Let us notice that this equation describes the classical spin-$\frac{3}{2}$ field (2.2) with mass M and momentum $p = -\mathrm{i}M\dot{x}$. Therefore, $|\dot{x};\nu\rangle$ obeys the Rarita–Schwinger equation (2.7)†

$$\hat{\dot{x}}|\dot{x};\mu\rangle = |\dot{x};\mu\rangle \qquad \sigma^\mu|\dot{x};\mu\rangle = 0. \tag{4.5}$$

The solution of the first relation has the form

$$|\dot{x};\mu\rangle = C_\mu|\dot{x}\rangle \tag{4.6}$$

† *These equations follow from the previous relation* after multiplication of both sides by σ_μ and $\dot{x}_\mu$

where the vector $|\dot{x}\rangle$ was defined in (4.1). From the second equation, using the identity $\sigma_\mu \hat{\dot{x}} \sigma_\nu |\dot{x};\, \nu\rangle = (\mathrm{i}\varepsilon_{\mu\nu\rho}\dot{x}_\rho - \hat{\dot{x}}\delta_{\mu\nu})|\dot{x};\, \nu\rangle = 0$ one gets the relation for the vector C_μ

$$C_\mu = -\mathrm{i}\varepsilon_{\mu\nu\rho}\dot{x}_\rho C_\nu.$$

After the decomposition of C_μ on real and imaginary parts

$$C = \frac{1}{\sqrt{2}}(e_1 + \mathrm{i}e_2) \tag{4.7}$$

the vectors e_1 and e_2 satisfy the same equations as before in (4.3). Two linear independent solutions of (4.3) give us two vectors C_μ differing from one another by non-essential factor. Thus, the spin factor for the spin-$\frac{3}{2}$ field is equal to

$$\Phi_{J=3/2}(P) = -\mathrm{i}\int_0^T \mathrm{d}t \left\langle \dot{x};\, \mu \left| \frac{\mathrm{d}}{\mathrm{d}t} \right| \dot{x};\, \mu \right\rangle = -\mathrm{i}\int_0^T \mathrm{d}t \left\langle \dot{x} \left| \frac{\mathrm{d}}{\mathrm{d}t} \right| \dot{x} \right\rangle + \tfrac{1}{2}\int_0^T \mathrm{d}t(e_1\dot{e}_2 - e_2\dot{e}_1)$$
$$= \tfrac{3}{2}\int_0^T \mathrm{d}t\, C(t).$$

The explicit form of the state (4.6) enables us to generalize the evaluation of the spin factor for higher spin fields. The crucial point is that for an arbitrary spin J the states $|\dot{x}, a\rangle$ entering into (3.9) are the solutions of the classical equation of motion (2.2) for the fields with spin J, momentum $p = -\mathrm{i}M\dot{x}$ and mass M.

Higher half-integer spins. For an arbitrary half-integer spin $J = \frac{1}{2} + Z$ the wave equations have the form [9] analogous to (4.5)

$$(\hat{\dot{x}} - 1)|\dot{x}, \mu_1 \ldots \mu_Z\rangle = \sigma^{\mu_1}|\dot{x}, \mu_1 \ldots \mu_Z\rangle = 0$$
$$|\dot{x}, \ldots \mu_i \ldots \mu_j \ldots\rangle = |\dot{x}, \ldots \mu_j \ldots \mu_i \ldots\rangle,\ |\dot{x}, \ldots \mu_i \ldots \mu_i \ldots\rangle = 0.$$

The normalized solution of these relations is

$$|\dot{x};\, \mu_1 \ldots \mu_Z\rangle = |\dot{x}\rangle C_{\mu_1} \ldots C_{\mu_Z} \tag{4.8}$$

where the vectors $|\dot{x}\rangle$ and C_μ were defined in (4.1) and (4.7), and zero trace of state is achieved due to the property $C^2 = \frac{1}{2}(e_1 + \mathrm{i}e_2)^2 = 0$. Thus, for fields with an arbitrary half-integer spin the spin factor is equal to

$$\Phi_{J=\frac{1}{2}+Z}(P) = -\mathrm{i}\int_0^T \mathrm{d}t \left\langle \dot{x};\, \mu_1 \ldots \mu_Z \left| \frac{\mathrm{d}}{\mathrm{d}t} \right| \dot{x};\, \mu_1 \ldots \mu_Z \right\rangle$$
$$= -\mathrm{i}\int_0^T \mathrm{d}t \left\langle \dot{x} \left| \frac{\mathrm{d}}{\mathrm{d}t} \right| \dot{x} \right\rangle + \frac{Z}{2}\int_0^T \mathrm{d}t(e_1\dot{e}_2 - e_2\dot{e}_1)$$
$$= J\int_0^T \mathrm{d}t\, C(t) \tag{4.9}$$

and is the product of the spin and the torsion of the path.

Higher integer spins. The classical fields with an arbitrary integer spin $J = Z$ are described by symmetric traceless real tensor fields $A_{\mu_1 \ldots \mu_J}$ and $F_{\mu_1 \ldots \mu_J}$ forming the state

$$|\dot{x}\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} A_{\mu_1 \ldots \mu_J} \\ F_{\mu_1 \ldots \mu_J} \end{pmatrix}$$

and satisfying the equations [9] analogous to (4.5)

$$\varepsilon_{\mu_1\nu\rho}\dot{x}_\nu A_{\rho\mu_2\ldots\mu_J} - F_{\mu_1\ldots\mu_J} = 0 \qquad \varepsilon_{\mu_1\nu\rho}\dot{x}_\nu F_{\rho\mu_2\ldots\mu_J} + A_{\mu_1\ldots\mu_J} = 0.$$

The system has two nontrivial solutions

$$A^{(1)}_{\mu_1\ldots\mu_J} = \frac{1}{\sqrt{2}}(C^*_{\mu_1}\ldots C^*_{\mu_J} + C_{\mu_1}\ldots C_{\mu_J})$$

$$F^{(1)}_{\mu_1\ldots\mu_J} = \frac{i}{\sqrt{2}}(C^*_{\mu_1}\ldots C^*_{\mu_J} - C_{\mu_1}\ldots C_{\mu_J})$$

and

$$A^{(2)}_{\mu_1\ldots\mu_J} = -F^{(1)}_{\mu_1\ldots\mu_J} \qquad F^{(2)}_{\mu_1\ldots\mu_J} = A^{(1)}_{\mu_1\ldots\mu_J}$$

which have the properties

$$A^{(a)}_{\mu_1\ldots\mu_J}A^{(b)}_{\mu_1\ldots\mu_J} = F^{(a)}_{\mu_1\ldots\mu_J}F^{(b)}_{\mu_1\ldots\mu_J} = \delta^{ab}$$

and form two orthonormal states

$$|\dot{x}, a; \mu_1\ldots\mu_J\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} A^{(a)}_{\mu_1\ldots\mu_J} \\ F^{(a)}_{\mu_1\ldots\mu_J}\end{pmatrix}. \tag{4.10}$$

Hence, for fields with an arbitrary integer spin J the spin factor is equal to

$$\begin{aligned}\Phi^{ab}_{J=Z}(P) &= -\mathrm{i}\int_0^T \mathrm{d}t\left\langle \dot{x}, a; \mu_1\ldots\mu_J\left|\frac{\mathrm{d}}{\mathrm{d}t}\right|\dot{x}, b; \mu_1\ldots\mu_J\right\rangle \\ &= -\frac{J}{2}(\sigma_2)^{ab}\int_0^T \mathrm{d}t(e_1\dot{e}_2 - e_2\dot{e}_1) \\ &= -J(\sigma_2)^{ab}\int_0^T \mathrm{d}t\, C(t).\end{aligned} \tag{4.11}$$

Thus, the eigenstates and spin factors entering into the expressions (3.16) for the propagator and the effective action are given by (4.8) and (4.9) for half-integer and by (4.10) and (4.11) for integer spins.

Returning to the Berry connection (3.13) we note that for half-integer spins $r = 1$ and $\mathscr{A}^{ab}_\mu(\dot{x})$ is the Abelian gauge field. In the simplest case of Dirac fermions ($K = 0$) the field strength (3.14) is given after substitution $\Pi_\pm(\dot{x}) = \frac{1}{2}(1 \pm (\dot{x}\cdot\sigma))$ by

$$\mathscr{F}_{\mu\nu}(\dot{x}) = -\frac{\mathrm{i}}{4\dot{x}^2}\mathrm{Tr}(\Pi_+(\dot{x})\sigma_\mu\Pi_-(\dot{x})\sigma_\nu - \Pi_+(\dot{x})\sigma_\nu\Pi_-(\dot{x})\sigma_\mu) = \frac{1}{2\dot{x}^2}\varepsilon_{\mu\nu\rho}\dot{x}_\rho$$

where σ_μ are Pauli matrices. One recognizes in this expression the strength of field created by monopole with charge $\frac{1}{2}$ placed at the point $\dot{x}_\mu = 0$. Comparing (3.12), (4.9) and (4.11) we notice that generalization of this result to arbitrary half-integer spins J can be achieved by changing of monopole charge from $\frac{1}{2}$ to J, but for integer spins J one has to replace monopole charge by matrix $(-J\sigma_2)$.

5. Parity of the effective action

The spin factor enters into the effective action (3.16) as $\mathrm{Tr}[T\exp(-\mathrm{i}\Phi(P))]$ and it is equal to $\exp(-\mathrm{i}J\int_0^T \mathrm{d}t\, C(t))$ for half integer spins and $2\cos(J\int_0^T \mathrm{d}t\, C(t))$ for integer spins. Such a difference between spin factors is a manifestation of the parity anomaly appearing in the effective action of odd dimensional fermions.

Before examining the parity of the effective action (3.16) we have to take care of ultraviolet divergences in the perturbation theory expansions of $W[A]$. In the path integral representation (3.2) it is the singularity of the integration measure $\int_0^\infty (\mathrm{d}T/T)\,\mathrm{e}^{-TM}$ and the matrix element $\int \mathrm{d}^3x\langle x|\,\mathrm{e}^{\mathrm{i}TH}|x\rangle$ in the limit $T\to 0$ that is the source of ultraviolet divergences. Hence, the summation over loops with vanishing length T contributes to ultraviolet divergences. To regularize the effective action, one uses the gauge-invariant Pauli–Villars regularization

$$W_{reg}[A] = -\log\det((D\cdot\Gamma)+\mathrm{i}M)+\log\det((D\cdot\Gamma)+\mathrm{i}M_{\mathrm{PV}})$$

with regulator mass $M_{\mathrm{PV}} \gg M$. After substitution of (3.16) into this expression the integration measure over T is replaced by $\int_0^\infty (\mathrm{d}T/T)\,\mathrm{e}^{-TM}(1-\mathrm{e}^{-TM_{\mathrm{PV}}})$ and has well defined behaviour in the limit $T\to 0$.

In the lattice approximation the regularized effective action reduces to

$$W[A] = \sum_P \exp(-ML(P))\,\mathrm{Tr}[T\exp(-\mathrm{i}\Phi(P))]\,\mathrm{Tr}\,P\exp\left(\mathrm{i}\oint_P \mathrm{d}x^\mu A_\mu(x)\right)$$

where summation is performed over all closed paths in the three-dimensional Euclidean spacetime. Under parity transformations defined as

$$P: \qquad x_\mu = (x_1, x_2, x_3) \to x_\mu^P = (-x_1, x_2, x_3)$$
$$A_\mu(x) \to A_\mu^P(x) = (-A_1(x), A_2(x), A_3(x))$$

the length L of the loop and P-ordered exponential are unchanged but the torsion $C(t)$ changes in sign. The spin factor being the only source of anomalous parity properties is even function of the torsion only for integer spins. Thus, the effective action of integer spin fields is scalar but the effective action of half-integer spin fields is transformed non-trivial and is decomposed for an arbitrary mass M into a sum of scalar and pseudoscalar contributions [15].

6. Conclusion

It is well known that quantum scalar field describes randomly walking spinless relativistic particles in the path-integral approach. The generalization to spinning particles is achieved by introducing the path depending 'spin factor' into the path integral for scalar particle [1]. It turns out that for particles with an arbitrary half-integer spin the spin factor coincides with the Abelian Berry phase and is equal to a product of the spin and the torsion of the path. For particles with integer spin the spin factor being the non-Abelian Berry phase is a product of the Pauli matrix, spin and the torsion of the path. Different properties of the spin factors for integer and half-integer spins are the manifestation of the parity anomaly appearing in the effective action of three-dimensional fermions.

The generalization of the above results to higher dimensions using methods [7] is straightforward.

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Appendix

Under rotations of the vector p_μ with angle ω_μ the field $\phi(p)$ is transformed in accordance with a reducible representation of the SO(3) group

$$\phi(p) \to \exp(\mathrm{i}\Sigma^\mu \omega_\mu)\phi(p)$$

where Σ^μ are generators of the representation. The Lorentz covariance of (2.2) implies that Σ^μ obey the equation

$$(p^\omega \cdot \Gamma) = U(p \cdot \Gamma)U^+ \qquad U = \exp(\mathrm{i}\Sigma^\mu \omega_\mu).$$

Replacing vectors p_μ^ω and p_μ by the tangent vectors $\dot{x}_\mu(t+\delta t)$ and $\dot{x}_\mu(t)$, respectively, one gets using (2.17)

$$\Pi_+(\dot{x}(t+\delta t)) = U\Pi_+(\dot{x}(t))U^+ \qquad U = \exp(\mathrm{i}\Sigma^\mu \omega_\mu) \tag{A1}$$

with ω_μ being the angle between the vectors $\dot{x}_\mu(t+\delta t)$ and $\dot{x}_\mu(t)$.

Let us denote the product of two projectors as follows

$$\Pi_+(\dot{x}(t+\delta t))\Pi_+(\dot{x}(t)) = V\Pi_+(\dot{x}(t)). \tag{A2}$$

Then matrix V satisfies the equation

$$\Pi_+(\dot{x}(t+\delta t))\Pi_+(\dot{x}(t))\Pi_+(\dot{x}(t+\delta t)) = V\Pi_+(\dot{x}(t))V^+ \tag{A3}$$

whose LHS after substitution of $\Pi_+(\dot{x}) = \Sigma_{a=1}^r |\dot{x}, a\rangle\langle\dot{x}, a|$ in the limit $\delta t \to 0$ is equal to $\Pi_+(\dot{x}(t+\delta t)) + \mathcal{O}(\delta t^2)$. Hence, up to $\mathcal{O}(\delta t^2)$ terms (A1) and (A3) coincide and one finds

$$V = U = 1 + \mathrm{i}\Sigma^\mu \omega_\mu + \mathcal{O}(\delta t^2) = 1 - \mathrm{i}\Sigma^\mu \varepsilon_{\mu\nu\rho}\dot{x}_\nu \ddot{x}_\rho \delta t + \mathcal{O}(\delta t^2). \tag{A4}$$

Combining (A2) and (A4) we obtain (3.15).

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